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Singular Instantons and Extra Dimensions

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Hawking and Turok (HT) have recently proposed that an open universe can be created from nothing. The instanton describing this process is singular, and therefore its validity has been subject to question. In particular, Vilenkin has shown that an instanton with the same singular structure as Hawking and Turok's would lead to the unsuppressed decay of flat space. However, Vilenkin's solution can be seen as the dimensional reduction of a five-dimensional nonsingular instanton. In this context the unsuppressed instability of flat space can be traded for metastability with a low decay rate, provided that the size of the extra dimension is large compared with the Planck scale. Implications for the HT model are discussed.

1. INTRODUCTION

We know that the universe is homogeneous, isotropic, and not too curved ($\Omega \sim 1$). Also, we know that at the time of last scattering there were small density perturbations of amplitude $\sim 10^{-5}$ on all scales larger than the horizon at that time. Standard inflationary models would explain why the universe is so close to homogeneous and isotropic and why there are small perturbations. At the same time, however, those models would predict that $\Omega \approx 1$ to very good accuracy. The reason is that in standard inflation, flatness and homogeneity are solved by the same mechanism, namely the accelerated expansion of the very early universe.

At present, observations are compatible with this prediction for Ω, but with increasing precision even a small departure from it would call for explanation. For that reason, there has been some interest in modified scenarios where the homogeneity problem is solved by quantum tunneling before

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a short period of slow-roll inflation starts. These are the so-called open inflation models. Quantum tunneling may occur in a first-order phase transition, where a supercooled inflationary false vacuum decays into true vacuum by nucleation of spherical bubbles. The interior of one of these bubbles looks like an open universe (with $\Omega < 1$). A short second period of slow-roll inflation inside the bubble would lead to our observable universe. However, the inflaton potential required in these models is rather special, since one part of it has to be suitable for tunneling and another part has to be suitable for slow roll. These two processes require the coexistence of two different mass scales in the same potential, which is rather unnatural.

More recently, Hawking and Turok have suggested that an open universe may be created from nothing, without the need of a false vacuum phase [1]. This is an interesting possibility since it would lead to open inflation without requiring a special form of the potential. The price to pay, however, is that the instanton describing this process has a singular boundary (which is timelike in the Lorentzian region). Nevertheless, since the Euclidean action of the instanton turns out to be integrable, Hawking and Turok were able to proceed formally, and used their instanton to assign probabilities to different open universes.

The use of singular instantons has met with some objections [2–4]. It has been argued [4] that information may flow in and out from the singularity, which would mean that predictions cannot be made in that spacetime. On closer examination, however, it turns out that the singularity behaves as a reflecting boundary for scalar and tensor cosmological perturbations [5, 6]. Hence, the Cauchy problem seems to be well posed and the model is well suited for the quantization of small perturbations and for comparison with observations. Even so, it is clear that singular instantons cannot be used without further justification. Indeed, Vilenkin [3] has shown that an instanton with the same singularity as Hawking and Turok's would lead to the immediate decay of flat space, in contradiction with observations. Hence, there is some question as to whether singular instantons, even if integrable, can be used to describe the creation of an open universe. However, the unsuppressed decay of flat spacetime is due to the fact that singular instantons can have an arbitrarily small size, so that their action is as small as desired. It is therefore possible that if the model has a length scale below which physics is different, the instability can be traded for metastability with a low decay rate. In ref. 11, I presented a nonsingular five-dimensional model where some of the nice properties of the singularity in four dimensions can be understood and where the unsuppressed decay of flat space is avoided. Flat space is metastable, but its decay rate is exponentially small provided that the size of the extra dimension is much larger than the Plack length.

2. DECAY OF FLAT SPACE

Let us first show that flat space with an extra dimension is gravitationally metastable.[2] It decays through the nucleation of bubbles of "nothing" which eat up spacetime as they expand.

The five-dimensional action for pure gravity is given by

$$S_E = -\frac{1}{16\pi G_5}\int \sqrt{\tilde{g}}\,\mathfrak{R}\; d^5x - \frac{1}{8\pi G_5}\int \sqrt{\tilde{\gamma}}\,\tilde{K}\; d^4\xi \tag{1}$$

where G_5 is the five-dimensional gravitational coupling, $\mathfrak{R}$ is the Ricci scalar, and the last term is the integral over the boundary of the trace of the extrinsic curvature $\tilde{K}$. The tilde distinguishes five-dimensional quantities from their four-dimensional counterparts, which we shall encounter in the next section.

Taking an $O(4) \times U(1)$ symmetric ansatz for the metric

$$d\tilde{s}^2 = d\tau^2 + R^2(\tau)\, dS^{(3)} + r^2(\tau)\, dy^2 \tag{2}$$

where $dS^{(3)} = (d\psi^2 + \sin^2\psi\, d\Omega_2^2)$ is the metric on the three-sphere and y is the coordinate in the fifth compact dimension, the equations of motion reduce to (see, e.g., refs. 8) $\ddot{X} = 2k$ and $r\ddot{R} - \dot{r}\dot{R} = 0$, where $X \equiv R^2$, $k = 1$ is the spatial curvature of the 3-sphere, and dots indicate derivative with respect to τ. The first equation indicates that R^2 is quadratic in τ. The second tells us that r is proportional to $\dot{R}$. The constant of proportionality is unimportant, since it can be reabsorbed in a redefinition of y. Thus, the general solution is given by

$$d\tilde{s}^2 = d\tau^2 + (\tau^2 + A^2)dS^{(3)} + \left(\frac{\tau^2}{\tau^2 + A^2}\right) dy^2 \tag{3}$$

In what follows, we shall take $A^2 > 0$.

The instanton (3) is perfectly regular. At $\tau = 0$ there is a "polar" coordinate singularity in the (τ, y) plane, but the manifold is smooth and has no conical singularity if we take coordinate range in the fifth dimension as $0 \leq y < 2\pi A$. The size of the extra dimension is zero at $\tau = 0$ and goes to the constant value A at large distances $\tau \rightarrow \infty$. Thus the instanton can be viewed as the direct product of a "cigar" times a three-sphere. The size of the three-sphere tends to a constant A at $\tau = 0$, and grows linearly with τ at large distances, as it would in flat space. Thus this instanton is asymptotically flat. In fact, (3) is nothing but the 5-dimensional Euclidean black hole.

The solution (3) is analogous to Coleman and De Luccia's instanton [9] describing the nucleation of a "true vacuum" bubble. The important difference is that here there is no "true vacuum" to speak of. The interior of the 3-

[2] This process was first discussed by Witten [7]; see also Dowker *et al.* Caldwell *et al.* [7].

sphere of radius A at $\tau = 0$ contains no spacetime: it is a bubble of "nothing." The evolution of the bubble after nucleation is given by the analytic continuation of (3) to the Lorentzian section. This is obtained by complexifying the angular coordinate $\psi \to (\pi/2) - i\hat{\psi}$, where $\hat{\psi}$ is real. With this, the 3-spheres become (2 + 1)-dimensional timelike hyperboloids. The bubble grows with constant proper acceleration A^{-1}, eating up spacetime as it expands.

The nucleation rate can be estimated as [9]

$$\Gamma \sim A^{-4} B^2 e^{-B} \tag{4}$$

where $B = S_E - S_E^{\text{flat}}$ is the difference between the action of our instanton minus the action of flat space. Since (3) is a vacuum solution, only the boundary term at infinity ($\tau \to \infty$) contributes to the action. (Clearly, there is no boundary term at $\tau = 0$, since the fifth dimension smoothly closes the manifold there.) This term can be expressed as the normal derivative of the volume of the boundary,

$$S_E = \frac{-1}{8\pi G_5} \int \partial_\tau \sqrt{\gamma}\, d^3S^{(3)}\, dy \tag{5}$$

where $\sqrt{\gamma} = R^3 r = A^2\tau + \tau^3$. The integral in (5) diverges in the limit $\tau \to \infty$, but this is remedied when we subtract the corresponding term for flat space. The boundary has the topology of $S^3 \times S^1$. This is the boundary of a flat space solution where R is proportional to the distance to the origin and $r =$ const. Thus, the trace of the extrinsic curvature is given by $3R^{-1}$ and we have

$$S_E^{\text{flat}} = \frac{-3}{8\pi G_5} \int \sqrt{\gamma} R^{-1}\, d^3S^{(3)}\, dy \tag{6}$$

In the limit $\tau \to \infty$ we obtain

$$B = \frac{\pi A^2}{8G} \tag{7}$$

where $G = G_5/(2\pi A)$ is Newton's constant in four dimensions.

Thus, we find that even though flat space is metastable, the decay rate can be comfortably small provided that the size A of the extra dimension is much larger than the Planck length. For instance, in the context of M-theory (see, e.g., refs. 10), this size is of order $10^2 lp$, and the rate (4) would be unobservably small, even if we multiply it by the whole volume of our past light cone.

Now, what is smooth in five dimensions may look singular in four. Let us now show that the solution given in the previons section can be cast as Vilenkin's singular instanton [3]. The singularity is of the same form as the one in Hawking and Turok's solution.

Compactifying the fifth dimension on a circle and using the ansatz [8]

$$\tilde{g}_{AB} = e^{2\kappa\phi/3} \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & e^{-2\kappa\phi} \end{pmatrix} \tag{8}$$

where $\kappa = (12\pi G)^{1/2}$, we can write the action (1) as

$$S_E = \int \sqrt{g} \left[\frac{1}{2} (\partial\phi)^2 - \frac{\mathcal{R}}{16\pi G} \right] d^4x - \frac{1}{8\pi G} \int \sqrt{\gamma}\, K\, d^3\xi \tag{9}$$

This coincides with the action used in ref. 3.

Again, we take an $O(4) \times U(1)$ symmetric ansatz for the metric $g_{\mu\nu}\, dx^\mu\, dx^\nu = d\sigma^2 + b^2(\sigma)\, dS^{(3)}$ and for $\phi = \phi(\sigma)$. With this ansatz, the field equations reduce to

$$\phi'' + 3 \frac{b'}{b} \phi' = 0 \tag{10}$$

$$\left(\frac{b'}{b} \right)^2 = \frac{4\pi G}{3} \phi'^2 + \frac{1}{b^2} \tag{11}$$

where primes stand for derivatives with respect to σ. From the four-dimensional point of view, the instanton (3) corresponds to

$$b = \tau^{1/2} (A^2 + \tau^2)^{1/2} \tag{12}$$

$$\phi = \frac{-3}{4\kappa} \ln \left(\frac{\tau^2}{A^2 + \tau^2} \right) \tag{13}$$

Using $\left| d\tau/d\sigma \right| = (A^2 + \tau^2)^{1/4} \tau^{-1/2}$, it is straightforward to check that (12)–(13) satisfy the field equations (10)–(11).

At large τ, we have $\sigma \approx \tau$. Therefore $b \approx \sigma$ and $\phi \approx C/(2\sigma^2)$, where

$$C = 3A^2/2\kappa \tag{14}$$

Near $\tau = 0$ we have $\tau^{3/2} \approx (3A^{1/2}/2)(\sigma - \sigma_f)$, where σ_f is a constant. Substituting in (12) we find that $\phi \approx -\kappa^{-1} \ln(\sigma - \sigma_f) +$ const and $b^3 \approx C\kappa(\sigma - \sigma_f)$. Therefore, the solution (3) looks exactly like Vilenkin's singular solution when viewed in four dimensions.

3. INFLATION

In order to find inflationary solutions, a potential must be added to the action (9). In the context of Kaluza–Klein theories, it is believed that the dilaton will be stabilized by an effective potential generated by quantum corrections. It is possible that this same potential may drive inflation in the appropriate range. In ref. 11, a five-dimensional model with a cosmological constant was considered. The five-dimensional instanton is just the 5-sphere, which is obviously smooth. With a suitable dimensional reduction, it looks like an instanton of the Hawking–Turok type. That particular model is not realistic from the phenomenological point of view. For one, the reduction to four dimensions is not motivated by the smallness of the extra dimension, since all dimensions are of the same size. However, it has the advantage of leading to explicit solutions. More realistic models are currently under investigation.

Although the arguments presented above seem to validate the use of singular instantons, some words of caution must be said. The smooth instanton presented in ref. 11 exists only when the value of inflaton field at the beginning of inflation takes a particular value (which makes the size of the fifth dimension equal to the radius of the 5-sphere). Otherwise, there will be a conical singularity at the point where the fifth dimension closes. The same would happen if we replace the cosmological constant in five dimensions by a general "inflaton" potential: the fifth dimension would only close smoothly for a particular value of ϕ at the beginning of inflation. Hence, it appears that these instantons cannot produce a range of values of the density parameter. This casts some doubt on the method used in ref. 1 to find the probability distribution for Ω. On the positive side, it is still true that one can obtain an open universe without requiring a special form of the potential. We should add that there are more "conventional" ways of obtaining a range of values of Ω in theories where one field undergoes a first-order phase transition and a second field is responsible for slow roll inflation inside the nucleated bubbles [12]. In such models, one finds that a range of values of Ω occurs inside of *each* nucleated bubble [13], and depending on parameters of the model, it is not unlikely for an observer to measure the density parameter in the range $(1 - \Omega)/\Omega \sim 1$ [14].

4. COSMOLOGICAL PERTURBATIONS AND CMB ANISOTROPIES

An interesting feature of the four-dimensional HT models is that linearized cosmological perturbations can be consistently quantized in spite of the timelike singularity [5]. This is because the singularity acts as a reflecting boundary for both scalar and tensor perturbations. This is perhaps not too surprising if we consider the scattering from the five-dimensional point of view, where there is no singularity and the Cauchy problem is well posed. Because the fifth dimension is homogeneous, the momentum in the fifth direction is a conserved quantity. Sending in a four-dimensional wave (that is, a wave which has no momentum in the fifth dimension) toward the singularity, the outcome to linear order can only be an outgoing wave with no momentum in the fifth dimension. Hence the scattering of four-dimensional cosmological perturbations is unitary, and no information flows in or out from the "singularity."[3]

In ref. 15, we considered the soluble model with potential

$$V(\phi) = \lambda e^{2\kappa\phi/3}$$

where λ is a constant, which leads to power law inflation with $a(t) \propto t^3$ (for $t \gg \lambda^{-1/4}$). This potential arises in the five-dimensional theory with a cosmological constant considered in ref. 11, but here we shall simply think of it as a four-dimensional model in its own right. The spectrum of cosmological density perturbations and gravity waves generated during inflation is found to be [15]

$$\langle |\mathcal{R}_c^p|^2 \rangle = \frac{24 G H_0^2}{(p^2 + 4)(p^2 + 1)} \tag{15}$$

$$\langle |h_{ij}^p h_p^{ij}| \rangle = \frac{64 G H_0^2}{(p^2 + 1)^2} \tag{16}$$

where $H_0^2 = (8\pi G/3)\lambda$ is the Hubble rate at the beginning of open inflation (where we have taken $\phi = 0$). In Eqs. (15) and (16), p is the eigenvalue of scalar and tensor harmonics on the open spacelike FRW surfaces ($0 < p < \infty$), where $p = 0$ corresponds to modes with wavelength comparable to the curvature scale,[4] $\mathcal{R}_c$ is the usual gauge-invariant curvature perturbation in the comoving hypersurface, and h_{ij} is the conformally rescaled transverse and traceless metric perturbation. Note that (15) and (16) are well behaved in the infrared limit $p \to 0$, and they behave as p^{-4} at large p. This corresponds to spectral indices $n_T = 0$ for scalar perturbations and $n_T = -1$ for tensors.

Phenomenologically one would like to have $n_T \approx 1$ in order to fit the scale-invariant spectrum of temperature anisotropies at low multipoles measured by COBE, and thus the model does not fit the data. But here we just consider this model a toy example, to show that no disastrous features arise in the spectrum due to the singular nature of the background spacetime. The spectrum of CMB temperature anisotropies which corresponds to the perturbations (15)–(16) can be obtained by solving the Boltzmann equation for photons through the time of equilibrium of matter and radiation and

[3] It should also be noted that problems will arise in the four-dimensional theory when we go beyond linear order, because an incoming graviton with sufficiently high momentum in four dimensions can decay into two "gravitons" with nonvanishing momenta in the fifth direction which do not belong to the spectrum of the four-dimensional theory. Clearly, the four-dimensional effective theory is still predictive below the momentum threshold corresponding to the compactification scale.

[4] There are no "supercurvature" modes in the spectrum for this particular model.

through recombination until the present time. We have done this [16] with the help of the CMBFAST code. The corresponding spectrum of temperature anisotropies is plotted in Fig. 1. Clearly, the singularity of the background does not imprint any disastrous features in the spectrum.

5. CONCLUSIONS

We have argued that singular instantons of the Hawking–Turok type are not necessarily disastrous, provided that the singularity can be resolved in a more fundamental theory. For instance, Vilenkin's singular instanton has a smooth analog in five dimensions. Also, the details of the fundamental theory need not be known in order to find the spectrum of cosmological perturbations, since linearized perturbations can be quantized in the singular background without ambiguity.

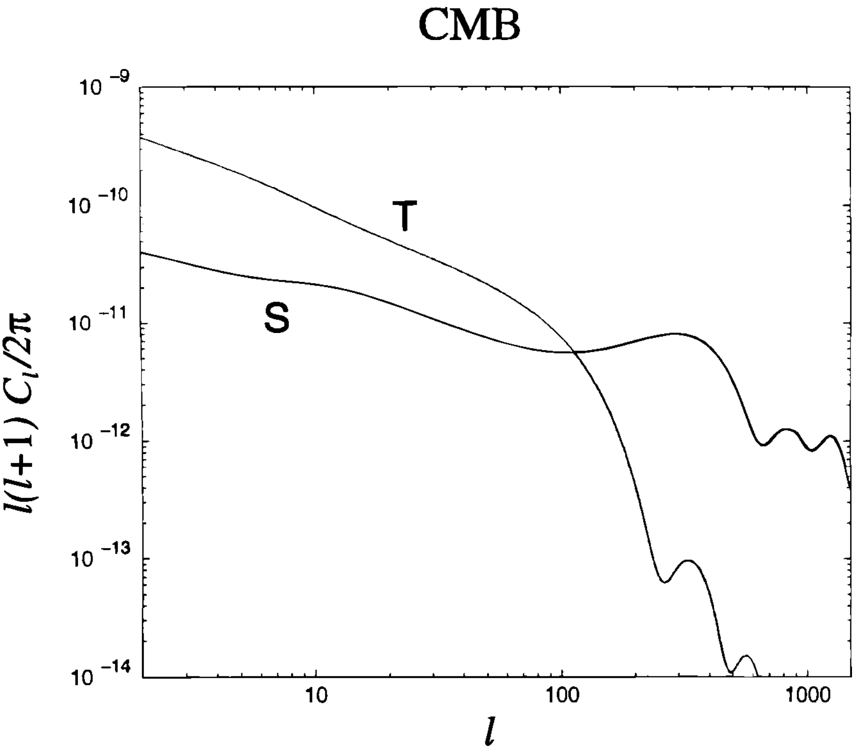

Fig. 1. CMB power spectrum, arbitrarily normalized, for the scalar (S) and tensor modes (T) for the Hawking–Turok model with exponential potential. The figure corresponds to $\Omega = .4$.

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